Math 308 Midterm #1, Autumn 2017

Name:

ID#:

Signature:

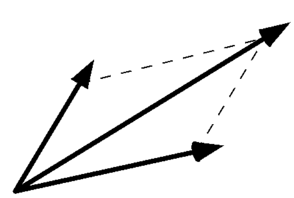
All work on this exam is my own.

Instructions.

- You are allowed a calculator and notesheet (handwritten, two-sided). Hand in your notesheet with your exam.
- Other notes, devices, etc are not allowed.
- Unless the problem says otherwise, **show your work** (including row operations if you row-reduce a matrix) and/or **explain your reasoning**. You may refer to any theorems, facts, etc, from class.
- All the questions can be solved using (at most) simple arithmetic. (If you find yourself doing complicated calculations, there might be an easier solution...)

1	/20
2	/20
3	/20
4	/10
5	/20

Good luck!



(1) (a) [5 pts each] Compute:

$$3\begin{bmatrix}1\\2\\1\end{bmatrix} - \tfrac{1}{2}\begin{bmatrix}0\\2\\2\end{bmatrix} = \begin{bmatrix}3\\5\\2\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 & -1\\2 & 3 & 0\end{bmatrix} \cdot \begin{bmatrix}1\\1\\-2\end{bmatrix} = 1\begin{bmatrix}0\\2\end{bmatrix} + 1\begin{bmatrix}1\\3\end{bmatrix} - 2\begin{bmatrix}-1\\0\end{bmatrix} = \begin{bmatrix}3\\5\end{bmatrix}$$

(b) [10 pts] Determine the general solution to the following system of equations:

$$\begin{aligned} z_1 + z_2 + 2z_3 \phantom{{}+z_4} &= 0 \\ 2z_1 + 2z_2 + 5z_3 + z_4 &= 1 \end{aligned}$$

Express your answer in vector form.

Solution. Convert to an augmented matrix and row-reduce:

$$\left[\begin{array}{cccc|c}1 & 1 & 2 & 0 & 0\\2 & 2 & 5 & 1 & 1\end{array}\right] \sim \left[\begin{array}{cccc|c}1 & 1 & 2 & 0 & 0\\0 & 0 & 1 & 1 & 1\end{array}\right].$$

This is in echelon form, with free variables z_2, z_4. So, set $z_2 = s_1$ and $z_4 = s_2$ and solve using back-substitution:

$$\begin{aligned} z_3 + z_4 = 1 &\quad\rightsquigarrow\quad z_3 = 1 - s_2. \\ z_1 + z_2 + 2z_3 = 0 &\quad\rightsquigarrow\quad z_1 = -2 - s_1 - 2s_2. \end{aligned}$$

In vector form, this gives:

$$\begin{bmatrix}z_1\\z_2\\z_3\\z_4\end{bmatrix} = \begin{bmatrix}-2 - s_1 - 2s_2\\s_1\\1 - s_2\\s_2\end{bmatrix} = \begin{bmatrix}-2\\0\\1\\0\end{bmatrix} + s_1\begin{bmatrix}-1\\1\\0\\0\end{bmatrix} + s_2\begin{bmatrix}-2\\0\\-1\\1\end{bmatrix}.$$

(2) (a) [10 pts] Does this set of vectors span $\mathbb{R}^3$?

$$\vec{\mathbf{v}}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \vec{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad \vec{\mathbf{v}}_3 = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}.$$

Solution 1. With three vectors in $\mathbb{R}^3$, we can compute the answer by row-reduction:

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the echelon form has a row of zeroes (i.e. a row without a pivot), the vectors do not span $\mathbb{R}^3$.

Solution 2. Some students noticed directly (or by row-reducing) that $\vec{\mathbf{v}}_3 = 3\vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2$. In other words, the set $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ is linearly dependent. By the Unifying Theorem / Big Theorem, (since $m = n = 3$) this means the vectors can't span $\mathbb{R}^3$.

(b) [10 pts] Does this set of vectors span $\mathbb{R}^5$?

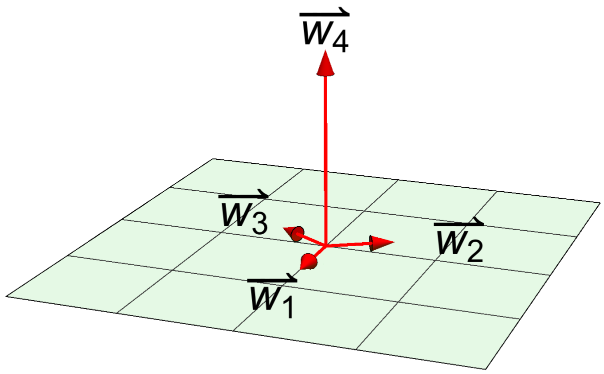
$$\vec{\mathbf{u}}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 4 \\ -2 \end{bmatrix}, \quad \vec{\mathbf{u}}_2 = \begin{bmatrix} 0 \\ 2 \\ 5 \\ 0 \\ -3 \end{bmatrix}, \quad \vec{\mathbf{u}}_3 = \begin{bmatrix} -1 \\ 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}.$$

Solution. No. By a theorem from class, we need at least five vectors to span $\mathbb{R}^5$.

(The long way to do this would be to row-reduce to echelon form, resulting in two rows of zeroes. But row-reduction is not necessary, in light of the argument above.)

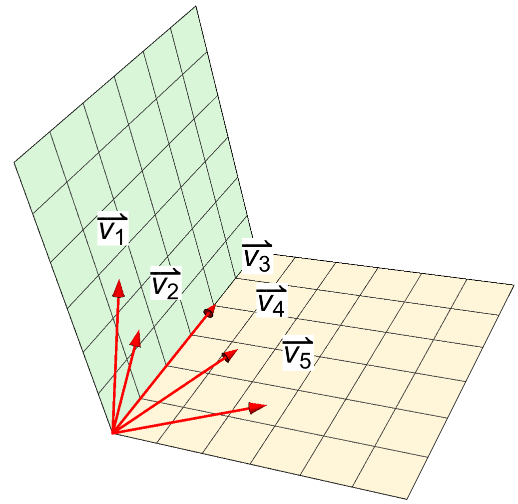
(3) Consider these arrangements of vectors in $\mathbb{R}^3$, then answer the questions below. **No justification is necessary.**

(I)



(Note: The plane contains $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3$.)

(II)



(Note: The left plane contains $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3$. The right plane contains $\vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5$.)

[6 pts] (a) From (I), with the vectors $\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3, \vec{\mathbf{w}}_4$:

Give any set of linearly independent vectors: $\left\{\vec{\mathbf{w}}_2\right\}$ or $\left\{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_3\right\}$ or $\left\{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_4\right\}$ or ...

Give any set of linearly dependent vectors: $\left\{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3\right\}$ or $\left\{\vec{\mathbf{w}}_1, \vec{\mathbf{w}}_2, \vec{\mathbf{w}}_3, \vec{\mathbf{w}}_4\right\}$.

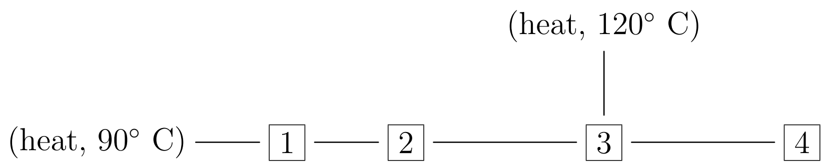
[8 pts] (b) From (II): Which of the following sets are linearly independent? Circle them:

$\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_3\}$ $\boxed{\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_3, \vec{\mathbf{v}}_5\}}$ $\boxed{\{\vec{\mathbf{v}}_3, \vec{\mathbf{v}}_4\}}$ $\{\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, \vec{\mathbf{v}}_4, \vec{\mathbf{v}}_5\}$

[6 pts] (c) Consider the 3×2 matrix $A = [\ \vec{\mathbf{w}}_1 \mid \vec{\mathbf{w}}_2\ ]$. Which of the following equations have solutions $\vec{\mathbf{x}}$? Circle them:

$\boxed{A\vec{\mathbf{x}} = \vec{\mathbf{w}}_3}$ $A\vec{\mathbf{x}} = \vec{\mathbf{w}}_4$ $\boxed{A\vec{\mathbf{x}} = \vec{\mathbf{0}}}$

(4) A chemical factory (Levinson's Linear Laboratory) has four tanks of liquid connected in a line, along with two heaters:



After a long time, the temperature t_i of the i-th tank will be the *average* temperature of the tanks and heaters connected to it. For example, t_1 should be the average of t_2 and 90.

(a) [5 pts] Write the system of equations you would use to determine t_1, t_2, t_3, t_4. You do **not** need to solve.

Solution. The equations are:

$$\begin{array}{lcr} t_1 = \frac{1}{2}(t_2 + 90) & \rightsquigarrow & t_1 - \frac{1}{2}t_2 = 45 \\ t_2 = \frac{1}{2}(t_1 + t_3) & \rightsquigarrow & -\frac{1}{2}t_1 + t_2 - \frac{1}{2}t_3 = 0 \\ t_3 = \frac{1}{3}(t_2 + 120 + t_4) & \rightsquigarrow & -\frac{1}{3}t_2 + t_3 - \frac{1}{3}t_4 = 120 \\ t_4 = t_3 & \rightsquigarrow & -t_3 + t_4 = 0. \end{array}$$

(b) [5 pts] Write the corresponding augmented matrix. You do **not** need to solve.

Solution.

$$\left[\begin{array}{cccc|c} 1 & -\frac{1}{2} & 0 & 0 & 45 \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{3} & 1 & -\frac{1}{3} & 40 \\ 0 & 0 & -1 & 1 & 0 \end{array}\right] \quad \text{(or equivalent).}$$

(5) In each of the following, either give an example or write "impossible".
No justification is necessary. [5 pts each]

(a) A set of vectors that spans $\mathbb{R}^2$ and is linearly dependent.

Solution. Many possible answers. Easiest: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$.

(b) A set of 4 vectors in $\mathbb{R}^3$ that do not span $\mathbb{R}^3$.

Solution. Many possible answers. Example: $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\0\\0\end{bmatrix}, \begin{bmatrix}3\\0\\0\end{bmatrix}, \begin{bmatrix}4\\0\\0\end{bmatrix}$.

(c) Three vectors that span $\mathbb{R}^3$ and satisfy the equation $\vec{\mathbf{v}}_1 - 2\vec{\mathbf{v}}_2 + \vec{\mathbf{v}}_3 = \vec{\mathbf{0}}$.

Solution. Impossible. (The equation forces the vectors to be linearly dependent. By the Unifying Theorem, three linearly dependent vectors in $\mathbb{R}^3$ cannot span $\mathbb{R}^3$.)

(d) An echelon system of equations in variables x_1, x_2, x_3 with free variable x_3. (Write out the equations.)

Solution. Many possible answers. Example:

$$\begin{aligned} x_1 + \quad x_2 + x_3 &= 1 \\ 2x_2 + x_3 &= 0. \end{aligned}$$

[**2 pts**] **Bonus.** What have you found easiest and hardest in Math 308?

Do you wish the pace was (circle): FASTER ABOUT THE SAME SLOWER

(OR: If you don't want to answer, draw a picture involving vectors.)